Scaling of Correlation f^ns and the critical exponent η

The most salient feature of a second-order phase transition is the divergence of correlation length as the critical point is approached. As we will discuss in detail soon, approaching from the ordered side of the transition, $\langle s(r)\, s(r')\rangle \sim m^2 + e^{-|r-r'|/\xi}$ where $m \sim (T_c - T)^{\beta}$, and $\xi \sim \dfrac{1}{(T-T_c)^{\nu}}$. Similarly, approaching from the disordered side, $\langle s(r)\, s(r')\rangle \sim e^{-|r-r'|/\xi}$.

One way to calculate correlation f^n is to add a 'source term' $-\sum_r h(r)\, s(r)$ in the Hamiltonian:

$$H = -J \sum_{\langle r r' \rangle} s(r)\, s(r') - \sum_r h(r)\, s(r)$$

and noticing that the connected correlation f^n $G(r_1, r_2)$

$\langle s(r_1)\, s(r_2)\rangle - \langle s(r_1)\rangle \langle s(r_2)\rangle$ is given by

$$= \left.\frac{\partial^2 \log Z}{\partial h(r_1)\, \partial h(r_2)}\right|_{h=0}$$

where $Z = \mathrm{Tr}\, e^{-H}$.

The renormalized Hamiltonian is given by

$$H' = -J' \sum_{\langle r r' \rangle} s(r)\, s(r') - \sum_{r'} h'(r')\, s'(r')$$

By definition of RG eigenvalue $h'(r') = b^{y_h} h(r)$.

RG transformation preserves the partition f^n,

therefore

$$Z'(h', t') = Z(h, t)$$

$$\Rightarrow \quad \frac{\partial^2 \log Z'(h', t')}{\partial h'(r'_1)\, \partial h'(r'_2)} = \frac{\partial^2 \log Z_1(h, t)}{\partial h'(r'_1)\, \partial h'(r'_2)}$$

Let's first consider the L.H.S. It is simply the correlation f^n of the renormalized Hamiltonian between spins at r'_1 and r'_2. Since the $|r'_1 - r'_2| = \frac{|r_1 - r_2|}{b}$, where r_1, r_2 denotes the locations of the pre-RG spins that were mapped to r'_1, r'_2, this correlation f^n is simply $G\left(\frac{r_1 - r_2}{b}, t'\right)$.

The RHS requires a bit more thought since we are taking the derivative of $Z(h, t)$ w.r.t. h', let's write it out explicitly:

$$Z(h,t) = e^{\left[J \sum_{\langle rr' \rangle} S(r) S(r') + \sum_r h(r) S(r) \right]}$$

We need to express this in terms of the coarse-grained (block) variable $h'(r')$. Assuming that $|r_1 - r_2| \gg$ size of a block, all $h(r)$ within a block map to $b^{-y_h} h'(r')$.

Therefore,

$$Z(h,t) = e^{\sum_{\langle rr' \rangle} J S(r) S(r') + b^{-y_h} \sum_{r'} h'(r') \sum_{r \in r'} S(r)}$$

Now we can take derivative w.r.t. $h'(r')$:

$$\frac{\partial^2 \log Z(h,t)}{\partial h'(r_1') \, \partial h'(r_2')} = b^{-2y_h} \left\langle \sum_{r \in r_1'} S(r) \sum_{r \in r_2'} S(r) \right\rangle_c$$

where 'c' in $\langle \ \rangle_c$ denotes connected correlator

i.e. $\langle S(r) S(r') \rangle_c = \langle S(r) S(r') \rangle - \langle S(r) \rangle \langle S(r') \rangle$.

Each block contains b^d pre-RG spins. After expanding $\left\langle \sum_{r \in r_1'} S(r) \sum_{r \in r_2'} S(r) \right\rangle_c$ one therefore obtains a sum of b^{2d} terms of the form $\langle S(r_i) \, S(r_j) \rangle_c$ where $r_i \in r_1'$ and $r_j \in r_2'$.

Since r'_1 and r'_2 are far-apart, we can assume that all of these b^{2d} terms equal $G(r_1 - r_2, t)$. Thus, LHS = RHS imply

$$G\left(\frac{r_1 - r_2}{b}, t'\right) = b^{2(d-y_h)} G(r_1 - r_2, t)$$

Since the RG eigenvalue of 't' (temperature like direction) is y_t, $t' = b^{y_t} t$. Denoting $r_1 - r_2 \equiv r$, one therefore finds

$$\boxed{G(r,t) = b^{-2(d-y_h)} G\left(\frac{r}{b}, t b^{y_t}\right)}$$

One way to 'solve' this relation is to follow the same procedure as we did for the scaling of free energy above. We repeat the RG n times, so that

$$G(r,t) = b^{-2n(d-y_h)} G\left(\frac{r}{b^n}, t b^{n y_t}\right)$$

and then set $b^{n y_t} t = t_0$. Thus,

$$G(r,t) = \left|\frac{t}{t_0}\right|^{\frac{2(d-y_h)}{y_t}} \psi\left(\frac{r}{|t/t_0|^{-1/y_t}}\right)$$

where ψ is a universal scaling f^n.

Close to the critical point (i.e. $t = h = 0$), we expect that $G(r,t) \sim e^{-r/\xi}$. Therefore we identify

$$\xi \sim |t|^{-1/y_t}$$

Defining a critical exponent ν as $\xi \sim |t|^{-\nu}$, we therefore obtain $\nu = 1/y_t$. Furthermore, at $t = 0$, we expect that $G(r)$ has a power-law dependence on r. This is confirmed by the above scaling form. As $t \to 0$, for $G(r,t)$ to be independent of t, if $\psi(x) \sim x^a$, then

$$2(d-y_h)/y_t + a/y_t = 0 \Rightarrow a = 2(y_h - d)$$

$$\Rightarrow G(r, t=0) \sim \frac{1}{r^{2(d-y_h)}}.$$

Defining a critical exponent η as $G(r,t=0) \sim \frac{1}{r^{d-2+\eta}} \Rightarrow \eta = d - 2y_h + 2.$

Therefore, all critical exponents $\alpha, \beta, \gamma, \delta, \nu, \eta$ are expressible in terms of y_h and y_t.

There is one catch however. We assumed that only relevant variables play important role in various scaling f^ns. There is a distinct possibility that the scaling f^ns may depend in a singular way on an irrelevant variable. Such a variable is called 'dangerously irrelevant', and we will see an example later. When that happens, then the naive relations between the thermodynamic exponents $(\alpha, \beta, \gamma, \delta)$ and the correlation f^n exponents (η, ν) may break down. For example, one may check that our above analysis implies a relation $\alpha = 2 - d\nu$, and this can break down in the presence of a dangerously irrelevant variable (recall: $\alpha = 2 - \frac{d}{y_t}$, $\nu = \frac{1}{y_t}$).

Critical exponents in terms of y_h, y_t

$$\alpha = 2 - d/y_t$$

$$\beta = (d - y_h)/y_t$$

$$\gamma = \frac{2y_h - d}{y_t}$$

$$\delta = \frac{y_h}{d - y_h}$$

$$\nu = 1/y_t$$

$$\eta = d - 2y_h + 2$$

As mentioned above, these relations hold assuming there are no dangerously irrelevant operators.

Rewriting of Scaling functions:

Above we obtained the following scaling forms in terms of y_h, y_t:

$$f_s(t,h) = \left|\frac{t}{t_0}\right|^{d/y_t} \phi\left(\frac{h/h_0}{|t/t_0|^{y_h/y_t}}\right)$$

$$G(r,t) = r^{-2(d-y_h)} \psi(r/t^{-1/y_t})$$

Using the definitions of critical exponents

$$\alpha = 2 - d/y_t, \quad \frac{y_h}{y_t} = \beta\delta,$$

$$2(d-y_h) = d-2+\eta, \quad y_t = 1/\nu \quad \text{and}$$

the correlation length $\xi \sim t^{-\nu}$, one can rewrite the above scaling forms as

$$f_s \sim |t|^{2-\alpha} \phi(h/t^{\beta\delta})$$

$$G(r,t) \sim \frac{1}{r^{d-2+\eta}} \psi(r/\xi)$$

In the presence of magnetic field,

$$G(r,t) \sim \frac{1}{r^{d-2+\eta}} \psi\left(r/\xi, \frac{h}{t^{\beta\delta}}\right)$$

Scaling Operators and Scaling Dimensions.

Above we saw that spin-spin correlations at the critical point decay as

$$\langle S(r)\, S(r')\rangle_c \sim \frac{1}{|r-r'|^{2(d-y_h)}}$$

They are determined by y_h because in the Hamiltonian contains the term $\sum_r h(r)\, S(r)$ and the RG eigenvalue of $h(r)$ is y_h.

One may now repeat the same argument for other correlation f^ns. For example consider the energy-energy correlation f^n $\langle E(r)\, E(r')\rangle_c$. To calculate this, one now adds a term $-\sum_r g(r)\, E(r)$ to the Hamiltonian and calculate $\dfrac{\partial^2 \log Z}{\partial g(r)\partial g(r')}$. Since the term $\sum_r g(r) E(r)$ is even under the Ising sym, under RG transformation $g(r') = b^{y_t} g(r)$. Repeating exactly same steps as that for the spin-spin correlations, one now obtains,

$$\langle E(r)\, E(r')\rangle_c \sim \frac{1}{|r-r'|^{2(d-y_t)}}$$

More generally, we define scaling operators ϕ_i which couple to u_i where u_i are eigenvectors to the RG transformation matrix T discussed earlier. If u_i have RG eigenvalue y_i, then

$$\langle \phi_i(r)\, \phi_i(r') \rangle \sim \frac{1}{|r-r'|^{2(d-y_i)}}.$$

The combination $d-y_i$ is called the 'scaling dimension' of the operator ϕ_i, and is typically denoted as x_i. Thus, $\langle \phi_i(r)\, \phi_i(r') \rangle \sim \frac{1}{|r-r'|^{2x_i}}$

Given a lattice operator $\hat{O}(r)$ which transforms in a certain way under various syms, one can decompose it as as sum of all scaling operators with those syms: $\hat{O}(r) \sim \sum_r c_i\, \phi_i(r)$

Thus, $\langle \hat{O}(r)\, \hat{O}(r') \rangle \sim \sum_{ij} \frac{c_i\, c_j}{|r-r'|^{x_i+x_j}}.$

Thus, the leading behavior is determined by the scaling operator that has non-zero overlap and the lowest scaling dimension.

Scaling dimension and Relevance / Irrelevance

Consider perturbing an RG fixed point with the term $u \sum_{r} O(r)$ where $O(r)$ has a scaling dim x

i.e. $\langle O(r)\, O(r') \rangle \sim \dfrac{1}{|r-r'|^{2x}}$. As just discussed, the scaling dimension x is related to the RG eigenvalue y of O via $y = d - x$. Therefore, this is a relevant perturbation if $d - x > 0$ i.e. $x < d$ (and irrelevant if $x > d$).

Therefore, the slower the decay of a correlation f^n of an operator, the more relevant it is.